# Errata and Corrections to *The Arithmetic of Dynamical Systems* 1st Edition

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**Acknowledgements Page vi**
Thanks to the following people who have sent me comments and corrections: Rob Benedetto, Xander Faber, Anthony Flatters, Liang-Chung Hsia, Patrick Ingram, Chong Gyu Lee.

**Page 4, line 25**
The sentence "In particular, we prove that $\mathcal{M}_2$ is an isomorphism to the affine plane $\mathbb{A}^2$." should read "In particular, we prove that $\mathcal{M}_2$ is a isomorphic to the affine plane $\mathbb{A}^2$."

**Page 4, line** $-6$
The book says that $\mathbb{P}^1(\mathbb{C}_p)$ is not Hausdorff, but it is clear that $\mathbb{P}^1(\mathbb{C}_p)$ is a Hausdorff space.

**Page 48, line** $-1$
$\mathbb{P}^1(k)$ should be $\mathbb{P}^N(k)$

**Page 49, line** $2$
$\mathbb{P}^1(k)$ should be $\mathbb{P}^N(k)$

**Page 49, line** $1$ **of example 2.6**
$\mathbb{P}^1(\mathbb{Q})$ should be $\mathbb{P}^3(\mathbb{Q})$

**Page 79, Exercise 2.24**
Add the assumption that the residue characteristic is not equal to 2.

**Page 84, line -9**
There is a missing absolute value symbol in the middle part of the displayed formula. It should read

$$\prod_{v\in M_K} \max_i\{|\alpha x_i|_v\}^{n_v} = \prod_{v\in M_K} |\alpha|_v^{n_v} \max_i\{|x_i|_v\}^{n_v} = \prod_{v\in M_K} \max_i\{|x_i|_v\}^{n_v}.$$

**Page 102, First displayed formula**
This is supposed to be the absolute height, so there should be a factor of $1/[K : \mathbb{Q}]$. Thus this line should read

$$h(P) = h(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} n_v \log \max\{|\alpha|_v, 1\},$$

**Page 103, Theorem 3.29, Equation (3.17)**
This is supposed to be the absolute height, so there should be a factor of $1/[K : \mathbb{Q}]$. Thus this line should read

$$\hat{h}_\phi(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} n_v \hat{\lambda}_{\phi,v}(\alpha) \qquad \text{for all } \alpha \in \mathbb{P}^1(K) \smallsetminus \{\infty\}.$$

**Page 105, Theorem 3.35**
The case $B = 0$ must be excluded.

**Page 105, Proof of Theorem 3.35**
Need to explain why the plane curves $G_1(X, Y) = B_1$ and $G_2(X, Y) = B_2$ have no common components.

**Page 108, Theorem 3.40 (Roth's theorem)**
Need to specify that $\alpha_v \notin K$, so the statement of the theorem should read "choose an algebraic number $\alpha_v \in \bar{K} \smallsetminus K$."

**Page 111, 2nd displayed equation**
The variable should be $z$, not $x$. Thus

$$\phi(z) = \frac{899z^2 - 2002z + 275}{33z^2 - 584z + 275}.$$

**Page 111, Proposition 3.46**
The rational map is supposed to have degree $d$. So it should read:

**Proposition 3.46.** *For all integers $N \geq 0$ and $d \geq 2$ there exists a rational map $\phi(z) \in \mathbb{Q}(z)$ of degree $d$ with the following properties*:

**Page 113, Table 3.1, Caption**
The caption should be $\mathcal{O}_\phi(1)$, not $\mathcal{O}_1(\phi)$.

**Page 114, Line $-8$**
It is not true that

$$\frac{1}{|a_n|^\epsilon} \geq \left|\frac{b_n}{a_n}\right|$$

is a consequence of $|a_n| \geq |b_n|^{1+\epsilon}$. So this needs to be adjusted. What is true is that

$$|a_n| \geq |b_n|^{1+\epsilon} \Longrightarrow \frac{1}{|a_n|^{\epsilon/(1+\epsilon)}} \geq \left|\frac{b_n}{a_n}\right|.$$

We may assume that $\epsilon \leq \frac{1}{2}$, which implies that

$$\frac{\epsilon}{1+\epsilon} \geq \frac{2}{3}\epsilon.$$

So assuming $\epsilon \leq \frac{1}{2}$, we have

$$|a_n| \geq |b_n|^{1+\epsilon} \Longrightarrow \frac{1}{|a_n|^{2\epsilon/3}} \geq \left|\frac{b_n}{a_n}\right|.$$

So can change the exponent on line $-8$ on page 114 to $\frac{2}{3}\epsilon$, and then line 3 on page 115 needs to be changed to

$$\log|a_n| \leq \frac{6}{\epsilon}\log(C_5^{-1}).$$

**Page 115, Line** 1
It should be $\geq$ instead of $=$.

**Page 115, Line** 3
It should be $C_5^{-1}$ instead of $C_5$, so the inequality should read

$$\log|a_n| \leq \frac{2}{\epsilon}\log(C_5^{-1}).$$

(However, see also the correction listed above for page 114, line $-8$.)

**Page 118, Equation (3.33)**
The "$f$" should be "$\phi$". Thus it should read

$$e_P(\phi^2) = e_P(\phi)e_{\phi(P)}(\phi) = e_0e_1 \leq d^2 - d.$$

**Page 121, First displayed equation**
The constants $C_7, C_8, C_9$ should be $C_8, C_9, C_{10}$, since $C_7$ was already used on the previous page. Thus the displayed equation should read

$$\begin{aligned}
\frac{1}{|a_n|^\epsilon} &\geq \rho(\phi^n(\alpha), \infty) && \text{from (3.34)},\\
&\geq C_8\rho\big(\phi^{n-m}(\alpha), \beta\big)^{\mathbf{e}_\infty(\phi^m)} && \text{from (3.36)},\\
&= C_8\left(\frac{b_{n-m}}{\sqrt{a_{n-m}^2 + b_{n-m}^2}}\right)^{\mathbf{e}_\infty(\phi^m)} && \text{definition of } \rho,
\end{aligned}$$

$$\geq \frac{C_9}{H\big(\phi^{n-m}(\alpha)\big)^{\mathsf{e}_\infty(\phi^m)}}$$ definition of height, where note that $b_{n-m} \neq 0$, since $\alpha$ is wandering and $\infty$ is periodic,

$$\geq \frac{C_{10}}{H\big(\phi^{n}(\alpha)\big)^{\mathsf{e}_\infty(\phi^m)/d^m}}$$ from Theorem 3.11,

$$\geq \frac{C_{10}}{H\big(\phi^{n}(\alpha)\big)^{\epsilon/6}}$$ from (3.35),

$$= \frac{C_{10}}{|a_n|^{\epsilon/6}}$$ since $H\big(\phi^n(\alpha)\big) = H(a_n/b_n) = |a_n|$.

**Page 128, footnote**
Add the assumption that the boundary of $U$ has measure 0. So the footnote should read: "Recall that a sequence of measures $\mu_i$ on a compact space $X$ *converges weakly* to $\mu$ if for every Borel-measurable set $U$ with boundary satisfying $\mu(\partial U) = 0$, the sequence of values $\mu_i(U)$ converges to $\mu(U)$ as $i \to \infty$.

**Page 130, Line $-9$ before Proposition 3.63**
The point $P$ has not been defined. So replace the sentence starting "Choose a prime ideal $\mathfrak{P}$..." with the following:

Let $P \in \mathrm{Per}_n^{**}(\phi)$ be a point of exact period $n$, and choose a prime ideal $\mathfrak{P}$ in $K_{\phi,n}^0(P)$ lying above $\mathfrak{p}$.

**Page 135, Exercise 3.3**
The quantity $D(P)$ is not defined. It is supposed to be the degree of the field of definition of $P$. So add the following at the beginning of this exercise:

For any $P \in \mathbb{P}^N(\bar{\mathbb{Q}})$, let $D(P) = [\mathbb{Q}(P) : \mathbb{Q}]$ be the degree of the field of definition of $P$.

**Page 137, Exercise 3.9**
Part (d) should certainly be marked as a hard problem, and possibly (c) should, too.

**Page 137, Exercise 3.13**
The stated result is true for any perfect field. For the first part of the proof, one assumes that $\phi^n(\sigma(P)) = \phi^m(P)$ for some $n \neq m$ and some $\sigma \in \mathrm{Gal}(\bar{K}/K)$ and shows that $P$ is preperiodic. Over number fields, I had in mind the following argument using the Galois invariance of the canonical height:

$$\begin{aligned} \hat{h}\big(\phi^n(\sigma(P))\big) &= \hat{h}\big(\phi^m(P)\big), \\ d^n\hat{h}(\sigma(P)) &= d^m\hat{h}(P), \\ d^n\hat{h}(P) &= d^m\hat{h}(P), \\ \hat{h}(P) &= 0, \end{aligned}$$

and hence $P$ is preperiodic. However, one can instead use a simple combinatorial argument and the fact that $\{\sigma^i P : i \geq 0\}$ is finite to prove this fact in general. Then the rest of the proof works over any perfect field.

**Page 138, Exercise 3.17**
Specify that the map $\phi$ has degree $d \geq 2$.

**Page 139, Exercise 3.21**
(b) The displayed equation

$$e^{n^2 \hat{h}_\phi(\alpha)} - \phi^n(2)$$

should be

$$e^{2^n \hat{h}_\phi(\alpha)} - \phi^n(2).$$

(In other words, the exponet should have $2^n$, not $n^2$.)
(c) "is the closest integer to $H^{n^2}$" should be "is the closest integer to $H^{2^n}$"

**page 139, Exercise 3.22**
$H^{n^2}$ should be $H^{d^n}$.

**page 139, Exercise 3.23**
In the displayed equation, the exponent should have $d^n$ instead of $n^2$. Thus it should read

$$\left| e^{d^n \hat{h}_\phi(\alpha)} + \frac{a}{d} - \phi^n(\alpha) \right| \leq \epsilon \qquad \text{for all } n \geq 0.$$

(Also move the $d^n$ to the other side of the $\hat{h}_\phi(\alpha)$, which is not an error, but makes the notation in Exercise 3.23 consistant with the notation in Exercise 3.21(b).)

**Page 140, Exercise 3.24(b)**
This is supposed to be the absolute height, so there should be a factor of $1/[K : \mathbb{Q}]$. Thus this line should read

$$\hat{h}_\phi(\alpha) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in M_K} n_v \hat{\lambda}_{\phi,v}(\alpha).$$

**Page 140, Exercise 3.32(a)**
The bound should be $\sqrt{4|B|/3}$ (missing absolute value sign).

**Page 150, Definition**
Reverse the order of the 2nd and 3rd bullet items to match the order of $\text{Per}_n$, $\text{Per}_n^*$, and $\text{Per}_n^{**}$.

**Page 151, Theorem 4.5(b) and proof**
Should not use $\lambda(P)$ to denote the multiplier at $P$, since that's not consistant with earlier notation. Use either $\lambda_P$ and $\lambda_P(\phi)$.

**Page 153, Line 2**
Insert "be" between "may" and "done".

**Page 158, Line −8**
Replace "primitive-$n$ periodic" with "primitive $n$-periodic". (Misplaced hyphen)

**Page 159**
Should not use $\lambda$ to denote a map, since too easily confused with the use of $\lambda$ as the multiplier of a map.

**Page 160, equation (4.17)**
The righthand side should be $\Phi^*_{n,P}$, i.e., add a star.

**Page 161, first paragraph**
It is not clear, *a priori*, that $\phi$ induces an automorphism of exact order $n$ on $Y_0(n)$. One needs to know that there is at least one value of $c$ for which $\phi_c(z) = z^2 + c$ has a point of primitive period $n$. The description of the bifurcation polynomials in Section 4.2.4 and the identification of their roots with a finite set of points of the Mandelbrot set gives the desired result.

**Page 174, Bottom of page**
The action of $\mathrm{PGL}_2$ on $\mathbb{Q}[\mathrm{Rat}_d]$ is defined as

$$R^f(\phi) = R(\phi^f).$$

However, this is a *left action*, i.e., $R^{fg} = (R^g)^f$, so the notation is confusing. It might be better to write the action as ${}^fR$. Otherwise add a note indicating that it is a left action.

**Page 180, line 3**
It should be $\lambda_P(\phi) \in \mathbb{C}$, not $\lambda_P(\phi) \in \mathbb{C}^*$, i.e., the multiplier may be 0.

**Page 187, lines 5 and 6**
Change $M_4$ to $\mathcal{M}_4$

**Page 187, lines 18 and 20**
Change $\boldsymbol{\sigma}_{d,N^*}$ to $\boldsymbol{\sigma}^*_{d,N}$

**Page 191, Proof of (a)**
Even assuming $\lambda_1\lambda_2 \ne 1$ and using (4.37), it does not follow that $\lambda_1 \ne 1$ and $\lambda_2 \ne 1$. However, all that we really neeed is that the fixed points associated to $\lambda_1$ and $\lambda_2$ are distinct. So replace the first two sentences of the proof of (a) with the following:

(a) For this part we assume that $\lambda_1\lambda_2 \ne 1$. If $\lambda_1 \ne \lambda_2$, then the fixed points associated to $\lambda_1$ and $\lambda_2$ are clearly distinct. But if $\lambda_1 = \lambda_2$, then our assumption implies that $\lambda_1$ and $\lambda_2$ are not equal to 1, so again the fixed points must be distinct. Hence we can find an element of $\mathrm{PGL}_2(\mathbb{C})$ that moves them to 0 and $\infty$, respectively.

**Page 192, Line $-10$**

The words "2-equivalent" should read "$\mathrm{PGL}_2$-equivalent." So the entire phrase reads:

Then Lemma 4.59(a)says that $\phi_1$ and $\phi_2$ are $\mathrm{PGL}_2$-equivalent to the function $(z^2 + \lambda_1 z)/(\lambda_2 z + 1)$,...

**Page 202, Definition**

The definition of cohomologous 1-cycles is not correct if $A$ is not abelian. The correct definition is that the 1-cocycles $g_1$ and $g_2$ are cohomologous if there is an $f \in A$ such that

$$g_{2,\sigma} = f g_{1,\sigma} \sigma(f^{-1}) \qquad \text{for all } \sigma \in G.$$

Note that if $A$ is abelian, then this is equivalent to $g_{1,\sigma}^{-1} g_{2,\sigma} = f\sigma(f^{-1})$ being a coboundary.

**Page 202, Remark 4.78**

Using the correct definition of cohomologous cycles, the indicated map is a well-defined injection of $\mathrm{Twist}(X/K)$ into $H^1(\mathrm{Gal}(\bar{K}/K), \mathrm{Aut}(X))$ even in the case that $\mathrm{Aut}(X)$ is nonabelian.

**Page 207, Definition of $G_\phi$**

It should be noted that $G_\phi$ is an open, and hence closed, subgroup of $\mathrm{Gal}(\bar{K}/K)$. To see this, let $E/K$ be a finite extension such that $\phi(z) \in E(z)$ and let $H = \mathrm{Gal}(\bar{K}/E)$. Then $H$ is an open neighborhood of the identity in $\mathrm{Gal}(\bar{K}/K)$, and for any $\sigma \in G_\phi$ and any $\tau \in H$ we have

$$\sigma\tau(\phi) = \sigma(\phi) = \phi^{g_\sigma},$$

so $\sigma H \subset G_\phi$.

**Page 208, Line $-5$**

It is not true that $\sigma \in G_\phi$ or that $G_\phi = \{1, \sigma\}$, since $\sigma \in \mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$, while $G_\phi \subset \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. This sentence should read:

This shows that any extension of $\sigma$ to $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is in $G_\phi$, so $G_\phi = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ and $K_\phi = \mathbb{Q}$.

**Page 215, first displayed equation**

The divisor $D'$ has degree $-2$, so it should be

$$D' = -(Q_1) - (Q_2).$$

**Page 221, Conjecture 4.97**

Patrick Ingram notes that this conjecture is not true. For example, the rational map $\phi(z) = z^2 + p^{-n}$ with $p \geq 3$ has minimal resultant that grows with $n$, while it has bad reduction only at $p$. (For example, if $n$ is even, then conjugation by $z \to z/p^{n/2}$ leads to a rational map with resultant $p^n$, which presumably is the minimal resultant.)

**Page 225, Exercise 4.6(b)**
Remove the $(z)$. Thus the displayed equation should read

$$\Phi^*_{n,\phi^p} = \phi^*_{n,\phi}\Phi^*_{np,\phi}.$$

**Page 225, Exercise 4.7**
$\lambda_\phi(\alpha)$ should be $\lambda_\alpha(\phi)$.

**Page 236, Exercise 4.42**
Remove the assumption that $\operatorname{Aut}(X)$ is abelian (see the correction to the definition of the cohomology set on page 202).

**Page 247, Remark 5.9**
Start this remark with the following assumption:

Let $r \in |K^*|$.

This is necessary, because if $r \notin |K^*|$, then one can have $\sum a_i(z-a)^i$ converging on $\bar{D}(a,r) = D(a,r)$ even though $|a_i|r^i \not\to 0$. Here is an example (shown to me by X. Faber) which might make a good exercise.

Let $K = \mathbb{C}_p$, $\alpha \in \mathbb{R} \smallsetminus \mathbb{Q}$, and $r = p^\alpha$. Let $\alpha_i \in \mathbb{Q}$ be a sequence monotonically increasing to $\alpha$. For each $i$, choose $a_i \in \mathbb{C}_p$ satisfying $|a_i| = p^{-i\alpha_i}$. Then $i(\alpha - \alpha_i) > 0$ for all $i$, so

$$|a_i|r^i = p^{i(\alpha-\alpha_i)} > 1.$$

On the other hand, if $z \in \bar{D}(a,r)$, then $|z-a| = p^\beta < r = p^\alpha$ for some $\beta \in \mathbb{Q}$. Since $\beta < \alpha$, we ahve

$$|a_i| \cdot |z-a|^i = p^{i(\beta-\alpha_i)} \sim p^{i(\beta-\alpha)} \to 0.$$

Thus $\sum a_i(z-z)^i$ converges on $\bar{D}(a,r)$.

**Page 302, Line $-10$**
$\bigwedge^\circ_{0,0}$ should be $\bigwedge^\circ_{0,1}$. So the sentence should read:

...attach one extra copy of $\bigwedge^\circ_{0,1}$ running vertically upward from the Gauss point $\xi_{0,1}$.

**Page 308, Line $-9$**
$\bigwedge^\circ_{0,0}$ should be $\bigwedge^\circ_{0,1}$. So the sentence should read:

We have seen that every point in $\bigwedge^\circ_{0,1}$ is attracted to $\xi_{0,0}$,...

**Page 311, Theorem 5.82(d)**
This is ungrammatical. It should read:

(d) Either the Julia set $\mathcal{J}^{\mathcal{B}}(\phi)$ is connected, or else it has infinitely many connected components.

**Page 364, Section 6.6**
The term *rigid Lattès map* is never defined! And many of the results in this section apply to all Lattès maps. This suggests that flexible Lattès maps are rigid. So probably Section 6.6 should be entitled "General Lattès Maps" and the term *rigid* should be reserved for Lattès maps associated to $P \mapsto [\alpha]P+T$ where $\alpha \notin \mathbb{Z}$. The reason that these are "rigid' is because they do not vary in a continuous family, whereas Lattès maps associated to $P \mapsto [m]P + T$ can be varied continuously by taking a familly of elliptic curves.

**Page 365, Theorem 6.57**
Specify that the map $E \to E/\Gamma$ is the natural projection $P \mapsto P \bmod \Gamma$.

**Page 381, Exercise 6.12**
The curve $E$ should be $y^2 = x^3 + b$, not $y^2 = x^3 + 1$.

**Page 383, Exercise 6.22(c)**
The sentence starting "More precisely" is actually the contrapositive of Proposition 6.55, not the converse. Replace that sentence with the following:

More precisely, if $\phi$ is a Lattès map fitting into a reduced Lattès diagram (6.37) and if $\phi^f$ has good reduction for some $f \in \mathrm{PGL}_2(K)$, does the elliptic curve $E$ also have good reduction?

**Page 384, Exercise 6.24(b)**
It should be "nontrivial element of $\Gamma$", not "nontrivial element of $\xi$".

**Page 384, Exercise 6.24(b)**
The quantity $e_P(\pi)$ should be the number of elements in the set, not the set itself. Thus it should read:

$$e_P(\pi) = \#\bigl\{\xi \in \Gamma : [\xi]P = P\bigr\}.$$

**Page 384, Exercise 6.24(c)**
Change "$T = 0$" to "$T = \mathcal{O}$".

**Page 393, Example 7.9, Line 2**
Change "Dehomogenizing" by "Homogenizing".

**Page 394, Theorem 7.10**
Parts of this theorem are false or not well-defined. If $\phi$ has degree 1, then $Z(\phi) = Z(\phi^{-1}) = \emptyset$, so $\ell_1$ and $\ell_2$ are not defined. And if $\phi$ is the identity map, then (c) is clearly false. So change the statement of the theorem to read:

Let $\phi : \mathbb{A}^N \to \mathbb{A}^N$ be a regular affine automorphism of degree at least 2.

**Page 395, Line** $-15$
Change "$\deg(\phi^n) = \deg(\phi)$" to "$\deg(\phi^n) = \deg(\phi)^n$". (Missing superscript $n$)

**Page 404, Definition in middle of page**
The prime divisors $W_1, \ldots, W_n$ are not *disjoint* as subsets of $V$, they are really the *distinct* irreducible subvarieties of $\phi^{-1}(W')$. So change "disjoint union of prime divisors" to "union of distinct prime divisors".

**Page 406, Line** $-10$
$\phi^* H$ should be $i^* H$, so the sentence should read "...then $i^* H$ is a very ample divisor on $V$."

**Page 438, Exercise 7.41b**
Specify that the Zariski closed subset is a proper subset, i.e., of all of $\mathbb{P}^{26}$.

**Page 438, Exercise 7.42c**
"such at" should be "such as".

**Page 459, References**
The article "Equidistribution and integral points for Drinfeld modules" that is attributed to H. Glockner (ArXiv:math.NT.0609120) should be attributed to D. Ghioca and T. Tucker.

**Page 484, Index**
The index entry for "exercise, hard" has a TEX error (caused by an extra backslash before textbf)

**Page 487, Index**
The index entry for "hard problem" has a TEX error (caused by an extra backslash before textbf)